

N-qubit entanglement via the J_y^2 -type collective interaction

D.L. Zhou^{a,b,*}, B. Zeng^c, J.S. Tang^d, Z. Xu^e, L. You^{a,b}

^a*School of Physics, Georgia Institute of Technology, Atlanta, GA 30332, USA*

^b*Institute of Theoretical Physics, The Chinese Academy of Sciences, Beijing, 100080, China*

^c*Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

^d*Department of Computer Science and Technology, Tsinghua University, Beijing, 100084, China*

^e*Department of Physics, Tsinghua University, Beijing, 100084, China*

Abstract

We investigate quantum correlations of the N -qubit states via a collective pseudo-spin interaction ($\propto J_y^2$) on arbitrary pure separable states for a given interval of time. Based on this dynamical generation of the N -qubit maximal entangled states, a quantum secret sharing protocol with N continuous classical secrets is developed.

Key words: quantum entanglement, quantum secret sharing

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1 Introduction

After more than ten years of active research and development in quantum information and quantum computation, quantum information science has become a major theme of contemporary physics research [1]. The understanding and characterization of quantum entanglement has emerged as a key fundamental issue. Despite much intense research efforts [2,3,4], we are still far from

* Corresponding author.

Email addresses: dz34@mail.gatech.edu (D.L. Zhou), zengbei@MIT.EDU (B. Zeng), tjs98@mail.tsinghua.edu.cn (J.S. Tang), zx-dmp@mail.tsinghua.edu.cn (Z. Xu), ly14@mail.gatech.edu (L. You).

forming a complete picture of multi-partite quantum correlations, especially in the limit of (large) N-qubit systems. In recent years, however, several studies have provided much insight into the controlled generation of a special class of N-qubit entangled states, the so-called maximally entangled, or N-GHZ states [5,6,7].

Mølmer and Sørensen [8] first discovered that N-GHZ states can be created with a J_y^2 -type collective interaction [9,10]. Their protocol as engineered in a linear ion trap, has led to the experimental demonstration of the largest Schrodinger cat state, an N-GHZ state of four trapped ions [11]. It was soon realized that the same interaction naturally occurs in a two component atomic condensate and can be utilized for the deterministic generation of entanglement atomic condensates [12].

Two somewhat puzzling points from these recent results stimulated the current study; 1) To generate an N-GHZ state with the J_y^2 -type interaction, the initial states were all product states of the same single qubit pure state, which is itself an eigenstate of $J_{x\perp}$, i.e. the collective spin component along a direction orthogonal to \hat{y} . 2) The N-GHZ protocol of J_y^2 shows an even/odd qubit number parity; it requires an extra single particle interaction term depending on whether N is even or odd, (see, for instance Ref. [8]).

This article provides an analytic approach to study the target maximally entangled N-GHZ states from the J_y^2 -type interaction when operated on any initial separable pure state. In addition to presenting a clear resolution of the above two mentioned puzzles from earlier studies of N-GHZ state creation, our result provides a general and convenient frame to discuss N-qubit entanglement.

This paper is organized as follows; First, we provide a general proof of a proposition using techniques developed for quantum error-correcting codes and cluster states [13,14,15]. We show that a J_y^2 -type interaction generates N-GHZ states for all N-qubit separable real pure states. Based on this extension, we provide a secret sharing protocol in which no single party can obtain his secret key without cooperations. We then suggest an optimal basis for projecting an arbitrary qubit onto the orthogonal plane to find the largest N-GHZ component as created by the J_y^2 -type interaction from an arbitrary N-qubit separable pure state; along the operation steps for the projection, we find that the system dynamics can be properly accounted for in terms of a set of complete and orthogonal N-GHZ states. We finally conclude with some commentary remarks.

2 Generating a N-GHZ state with a J_y^2 -type interaction from an unknown separable real state

Using the geometrical Bloch sphere representation of a qubit, an arbitrary pure state is represented by the vector from origin to the point (θ, ϕ) on the sphere. The special class of real separable pure states

$$|\psi\rangle = \prod_{k=1}^N \left(\cos \frac{\theta_k}{2} |0\rangle^{(k)} + \sin \frac{\theta_k}{2} |1\rangle^{(k)} \right), \quad (1)$$

corresponds to all N-qubit Bloch vectors lie in the z - x plane (as in Fig. 1 for a real qubit), with N unknown real parameters $\{\theta_i\}$ $i = 1, \dots, N$.

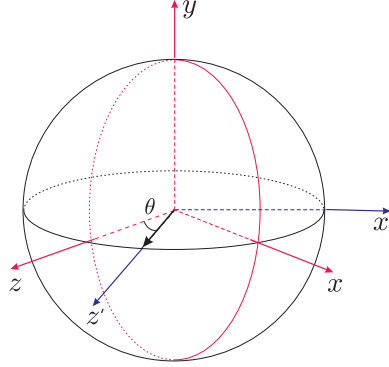


Fig. 1. An arbitrary pure qubit state in the equator z - x plane of the north-south axis \hat{y} (of \vec{J}_y), and the corresponding locally rotated coordinate (x', y, z') .

To generate a maximally entangled N-GHZ state from the sparable pure state (1), the most straightforward approach would be to invoke the unitary transformation

$$U_N = \frac{1}{\sqrt{2}} \left(I + i \prod_{k=1}^N \sigma_y^{(k)} \right). \quad (2)$$

This is difficult to realize for N-qubit physical systems, however, for it needs N -particle interactions. The J_y^2 -type interaction, on the other hand, involves only binary two-body interactions, and can be engineered in several known physical systems [8,11,12]. We now prove the following **proposition**: The unitary operator

$$S = \prod_{i,j=1, i < j}^N S_{ij}, \quad (3)$$

with

$$S_{ij} = (I + \sigma_y^{(i)} + \sigma_y^{(j)} - \sigma_y^{(i)} \sigma_y^{(j)})/2, \quad (4)$$

transforms a pure separable N-qubit real state $|\psi\rangle$ (1) into a maximally entangled N-GHZ state $|\psi_M\rangle$, i.e. $|\psi_M\rangle = S|\psi\rangle$.

Before proving our proposition, we note the following structure of a qubit state. As shown in Fig. 1, we define new spin matrices with respect to the locally rotated axis

$$\sigma_{z'}^{(i)} = \sigma_z^{(i)} \cos \theta_i + \sigma_x^{(i)} \sin \theta_i, \quad (5)$$

$$\sigma_{x'}^{(i)} = -\sigma_z^{(i)} \sin \theta_i + \sigma_x^{(i)} \cos \theta_i. \quad (6)$$

Clearly, the eigenstates of the $\sigma_{z'}^{(i)}$ with eigenvalues 1 and -1 are given by

$$|0\rangle_{z'}^{(i)} = \cos \frac{\theta_i}{2} |0\rangle^{(i)} + \sin \frac{\theta_i}{2} |1\rangle^{(i)}, \quad (7)$$

$$|1\rangle_{z'}^{(i)} = \sigma_{x'}^{(i)} |0\rangle_{z'}^{(i)} = \cos \frac{\theta_i}{2} |1\rangle^{(i)} - \sin \frac{\theta_i}{2} |0\rangle^{(i)}. \quad (8)$$

Similarly, the eigenstates of $\sigma_{x'}^{(i)}$ are

$$|0\rangle_{x'}^{(i)} = \frac{1}{\sqrt{2}}(|0\rangle_{z'}^{(i)} + |1\rangle_{z'}^{(i)}) = \cos \frac{\theta_i + \frac{\pi}{2}}{2} |0\rangle^{(i)} + \sin \frac{\theta_i + \frac{\pi}{2}}{2} |1\rangle^{(i)}, \quad (9)$$

$$|1\rangle_{x'}^{(i)} = \sigma_y^{(i)} |0\rangle_{x'}^{(i)} = -i \sin \frac{\theta_i + \frac{\pi}{2}}{2} |0\rangle^{(i)} + i \cos \frac{\theta_i + \frac{\pi}{2}}{2} |1\rangle^{(i)}. \quad (10)$$

It is easy to check that $\{\sigma_{x'}^{(i)}, \sigma_y^{(i)}, \sigma_{z'}^{(i)}\}$ satisfy all the commutation relations of Pauli operator, i.e. they form a group of Pauli matrices of a spin 1/2 operator.

With the above redefinition ($|\psi\rangle = \sigma_{z'}^{(i)} |\psi\rangle$), our proposition is equivalent to $|\psi_M\rangle = S \prod_{i=1}^N |0\rangle_{z'}^{(i)}$, or

$$|\psi_M\rangle = S \sigma_{z'}^{(i)} S^\dagger |\psi_M\rangle. \quad (11)$$

An easy calculation leads to

$$\begin{aligned} S_{ij} \sigma_{z'}^{(i)} S_{ij}^\dagger &= \sigma_{z'}^{(i)} \otimes \sigma_y^{(j)}, \\ S_{ij} \sigma_{z'}^{(j)} S_{ij}^\dagger &= \sigma_y^{(i)} \otimes \sigma_{z'}^{(j)}, \end{aligned} \quad (12)$$

and

$$\begin{aligned}
S_{ij}\sigma_{z'}^{(k)}S_{ij}^\dagger &= \sigma_{z'}^{(k)}, & k \neq i, j, \\
S_{ij}\sigma_y^{(k)}S_{ij}^\dagger &= \sigma_y^{(k)}, & \forall k = 1, \dots, N.
\end{aligned} \tag{13}$$

Thus we get

$$S\sigma_{z'}^{(i)}S^\dagger = \sigma_{z'}^{(i)} \prod_{j=1, j \neq i}^N \sigma_y^{(j)}, \tag{14}$$

i.e.

$$|\psi_M\rangle = \sigma_{z'}^{(i)} \prod_{j=1, j \neq i}^N \sigma_y^{(j)} |\psi_M\rangle. \tag{15}$$

The above N equations uniquely determines the state $|\psi_M\rangle$. We note that for every different pair $\{i, i'\}$,

$$\left(\sigma_{z'}^{(i)} \prod_{j \neq i} \sigma_y^{(j)} \right) \left(\sigma_{z'}^{(i')} \prod_{j' \neq i'} \sigma_y^{(j')} \right) = \sigma_{x'}^{(i)} \otimes \sigma_{x'}^{(i')}. \tag{16}$$

Thus, the state $|\Psi_M\rangle$ is limited to a two dimensional Hilbert space and can be expanded according to

$$|\psi_M\rangle = \alpha \prod_{i=1}^N |0\rangle_{x'}^{(i)} + \beta \prod_{i=1}^N |1\rangle_{x'}^{(i)}. \tag{17}$$

The coefficients α and β can be determined from

$$\sigma_{z'}^{(1)} \prod_{j=2}^N \sigma_y^{(j)} |\psi_M\rangle = |\psi_M\rangle, \tag{18}$$

and this leaves us with the explicit expression for $|\psi_M\rangle$,

$$|\psi_M\rangle = \frac{1}{\sqrt{2}} \left(\prod_{i=1}^N |0\rangle_{x'}^{(i)} + i \prod_{i=1}^N |1\rangle_{x'}^{(i)} \right), \tag{19}$$

which is a maximally entangled state. This completes our proof.

We now construct an equivalent Hamiltonian corresponding to the unitary operation Eq. (3). We note that

$$S_{jk} = \exp \left[-i\pi/4 + i\pi\sigma_y^{(j)}/4 + i\pi\sigma_y^{(k)}/4 - i\pi\sigma_y^{(j)}\sigma_y^{(k)}/4 \right],$$

which leads to

$$S = \exp\left(-iN(N-1)\frac{\pi}{4}\right) \exp\left(i(N-1)\frac{\pi}{4} \sum_{j=1}^N \sigma_y^{(j)}\right) \\ \times \exp\left(-i\frac{\pi}{4} \sum_{j,k=1, j < k}^N \sigma_y^{(j)} \sigma_y^{(k)}\right), \quad (20)$$

i.e. the unitary time evolution from an interaction Hamiltonian

$$H_I \propto \sum_{i,j=1, i < j}^N \sigma_y^{(i)} \sigma_y^{(j)} \sim J_y^2, \quad (21)$$

where $J_y = \sum_{i=1}^N \sigma_y^{(i)}/2$ is the y component of the total spin of the system and \sim means apart from a constant. It is well-known that the interaction Hamiltonian uJ_y^2 can be realized in many physical systems, such as trapped ions [8] and Bose-Einstein condensed atoms [12].

3 A quantum secret sharing protocol

Before extending the above discussion to arbitrary pure separable initial state, we want to comment on the maximally correlated nature of state (19) despite of the unknown θ_i 's. A simple criticism could naively point to the maximally entangled state (19) as a mathematical formula because its basis states [Eqs. (9) and (10)] are unknown. If θ_i were known, the initial state (1) becomes essentially the same as in the previous works [8,12], apart from local unitary transformations (basis rotations with known angles θ_i for every qubits). So what is new in this work? To address this important question, we have developed a quantum secret sharing protocol [16,17,18] to reveal the powerful multi-party quantum correlations of the state (19) irrespective of whether the basis states of the constituent qubits are known or not. The naive criticism arises due to a general lack of adequate understanding for multi-party entanglement, or more explicitly the lack of a reasonable multi-party entanglement measure. If there were such a measure, the entanglement of (19) is obviously unaffected by the θ_i 's.

Our quantum secret sharing protocol involves a queue $(\theta_1, \theta_2, \dots, \theta_N)$. The unknown angles θ_i 's can be encoded into the N-qubit initial state (1), the unitary operation (3) is then affected to create the maximally entangled state (19). Giving the j -th qubit to the j -th party, we accomplish a quantum secret sharing whereby no individual parties can obtain any information about the

queue alone, but the N parties can cooperate to uncover the complete queue. In fact, in this simple protocol, each individual party cannot even obtain its own key θ_j alone, because the reduced state for the j -th qubit is completely mixed

$$\rho_i^{(1)} = \text{Tr}_{1,2,\dots,j-1,j+1,\dots,N}(|\psi_M\rangle\langle\psi_M|) = \frac{1}{2}I. \quad (22)$$

To uncover the queue cooperatively, the N parties can simply execute the inverse of the operation (3). Each of the parties can then find its θ_j by local measurements in the initial real separable pure state (1).

This protocol shows that the maximally entangled state (19) remains useful due to the strong underlying N -qubit correlations despite of the unknown parameters θ_j 's. Our present work is therefore different from earlier ones [8,12] where only initial states with known θ_i ($\theta_i \equiv 0$) were discussed. Incidentally, we note that the above protocol for quantum secret sharing is not a threshold scheme [17] because it turns out that any m ($1 < m < N$) parties can work together to determine their own keys up to two possible choices without needing the other $N - m$ qubits. For example, in the case of $m = 2$, the reduced density matrix for two parties i and j becomes

$$\rho^{(ij)} = \frac{1}{2}|0\rangle_{x'}^{(i)}|0\rangle_{x'}^{(j)}\langle 0| \otimes |0\rangle_{x'}^{(j)}|0\rangle_{x'}^{(i)}\langle 0| + \frac{1}{2}|1\rangle_{x'}^{(i)}|1\rangle_{x'}^{(j)}\langle 1| \otimes |1\rangle_{x'}^{(j)}|1\rangle_{x'}^{(i)}\langle 1|, \quad (23)$$

which can be completely determined by measurements performed by parties i and j . It is worth emphasizing that the above decomposition of the reduced density matrix is unique since it is not only an eigen-decomposition but also a decomposition with separable state components. However, we cannot distinguish the component $|0\rangle_{x'}^{(i)}|0\rangle_{x'}^{(j)}$ from $|1\rangle_{x'}^{(i)}|1\rangle_{x'}^{(j)}$, thus we have two possible choices for the parameter pair (θ_i, θ_j) . This discussion for $m = 2$ remains valid for cases of $2 < m < N$, *i.e.* m parties can determine their secret keys θ s upto two possible alternative choices.

Before investigating the entangling dynamics of the J_y^2 -type interaction for more general N -qubit initial states, we hope to contrast our secret sharing protocol with several previously known schemes [16,17,18]. An obvious difference for our scheme is the need of many copies of identical quantum states. Previous protocols, on the other hand, usually require only a single copy of quantum state. However, the secret being shared by the multi-party through our protocol are unknown continuous classical variables, rather than basis states $|0\rangle$ or $|1\rangle$ of each qubit. To faithfully reconstruct a continuous variable from a qubit state always requires many identical copies. Furthermore, despite of a large number of copies of quantum states as required for our protocol, nothing about the shared secret can be obtained locally. This thus constitutes

a legitimate secret sharing protocol based on the quantum correlations of the N-GHZ type states with unknown parameters.

4 N-qubit entanglement with a J_y^2 -type interaction from an arbitrary pure separable state

The proposition established earlier enables a clear picture for the generation of N-GHZ states from an arbitrary separable initial pure state using an interaction $\sim J_y^2$. We now consider the more general initial separable pure state of N qubits

$$|\psi\rangle = \prod_{k=1}^N |\psi\rangle^{(k)}, \quad (24)$$

with

$$|\psi\rangle^{(k)} = \cos \frac{\theta_k}{2} |0\rangle^{(k)} + \sin \frac{\theta_k}{2} e^{i\phi_k} |1\rangle^{(k)}, \quad (25)$$

where all single qubit states are now specified by (θ_j, ϕ_j) as illustrated in Fig. 2. Clearly, the arbitrary wavefunction $|\psi\rangle$ can be expanded into the product basis $|0\rangle_z^{(i)}$ and $|1\rangle_z^{(i)}$ of the N-qubits. Since each of the product basis state, e.g. a state $\underbrace{|0\rangle_z^{(i_1)} \dots |0\rangle_z^{(i_m)}}_m \underbrace{|1\rangle_z^{(j_1)} \dots |1\rangle_z^{(j_{N-m})}}_{N-m}$ containing m -0s and $(N-m)$ -1s,

evolves into its corresponding maximally entangled N-GHZ state, this expansion enables a description of the N-qubit wavefunction in terms of a complete orthonormal N-GHZ basis.

To characterize the entanglement properties of the above N-qubit state, it is desirable to find the maximum coefficient into one arbitrary N-GHZ state [19]. The simple geometrical Bloch sphere representation might naively lead us to a decomposition of the state (θ_j, ϕ_j) , into a component in the orthogonal z - x plane, plus a parallel component along the y -axis. Unfortunately, this is incorrect because points on the Bloch sphere are spinors, not real vectors. Two points on the opposite side of the sphere, correspond to two vectors, not in (anti-)parallel, but in fact orthogonal. Perhaps somewhat strange, any state (θ_j, ϕ_j) can be decomposed into two orthogonal components along the opposite directions of a line through the origin.

We now study the optimal decomposition of state (24) with orthogonal states in the z - x plane (along the z^+ and z^- axis). We assume the optimal orthogonal states for k -th qubit are

$$|\phi^+\rangle^{(k)} = \cos \frac{\eta_k}{2} |0\rangle^{(k)} + \sin \frac{\eta_k}{2} |1\rangle^{(k)}, \quad (26)$$

$$|\phi^-\rangle^{(k)} = -\sin \frac{\eta_k}{2} |0\rangle^{(k)} + \cos \frac{\eta_k}{2} |1\rangle^{(k)}. \quad (27)$$

The initial state can then be expanded according to

$$\begin{aligned} |\psi\rangle &= \prod_{k=1}^N \sum_{s_k=+,-} {}^{(k)}\langle \phi^{s_k} | \psi \rangle^{(k)} |\phi^{s_k}\rangle^{(k)} \\ &= \sum_{\{s_k\}} \left(\prod_{k=1}^N {}^{(k)}\langle \phi^{s_k} | \psi \rangle^{(k)} \right) \prod_{k=1}^N |\phi^{s_k}\rangle^{(k)}. \end{aligned} \quad (28)$$

Following this decomposition, the final state after the evolution by the J_y^2 -type interaction becomes

$$|\psi_F\rangle = \sum_{\{s_k\}} \left(\prod_{k=1}^N {}^{(k)}\langle \phi^{s_k} | \psi \rangle^{(k)} \right) S \prod_{k=1}^N |\phi^{s_k}\rangle^{(k)}. \quad (29)$$

According to our proposition, $S \prod_{k=1}^N |\phi^{s_k}\rangle^{(k)}$ constitutes a maximally entangled orthonormal basis. Thus, equation (29) provides an alternative superposition of the 2^n orthogonal maximum entanglement states.

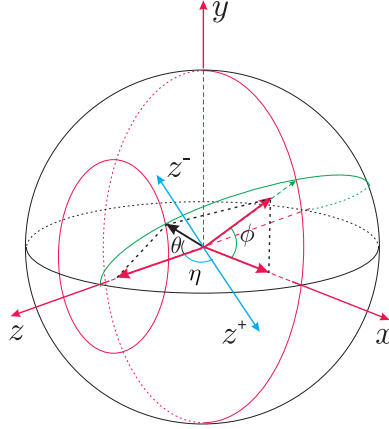


Fig. 2. An arbitrary qubit state can be decomposed into projections along orthogonal states in the equator z - x plane.

The optimal decomposition is found by maximizing the probability of one particular N-GHZ state basis state,

$$\begin{aligned} p(\eta_k) &\equiv \left| {}^{(k)}\langle \phi^+ | \psi \rangle^{(k)} \right|^2 \\ &= (1 + \cos \theta_k \cos \eta_k + \sin \theta_k \cos \phi_k \sin \eta_k) / 2, \end{aligned} \quad (30)$$

where $\eta_k \in (-\pi, \pi]$. This maximum is then specified by

$$\frac{\partial p(\eta_k)}{\partial \eta_k} = 0, \quad (31)$$

$$\frac{\partial^2 p(\eta_k)}{\partial \eta_k^2} < 0, \quad (32)$$

which gives

$$\tan \eta_k = \tan \theta_k \cos \phi_k, \quad (33)$$

$$\sin \eta_k \cos \phi_k > 0. \quad (34)$$

These conditions determine η_k completely. The solution actually has a simple geometrical interpretation: the projection of $\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ onto the z - x plane is $(\cos \theta, \sin \theta \cos \phi)$, which gives the angle defined by equation (33). The maximal probability becomes the product of the following for every qubit

$$p^{\max}(\eta_k) = \frac{1}{2} \left(1 + \sqrt{1 - \sin^2 \theta_k \sin^2 \phi_k} \right). \quad (35)$$

In order to make a conclusive statement for NPT entanglement of the N -qubits, $\prod_{k=1}^N p^{\max}(\eta_k)$ has to be larger than $1/2$ [19]. As a special case, we consider Bose condensed atoms in the same single qubit state

$$|\psi\rangle = \left(\cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle \right)^N. \quad (36)$$

The maximum probability then becomes $(1 + \sqrt{1 - \sin^2 \theta \sin^2 \phi})^N / 2^N$, which is illustrated on the Bloch sphere as in Fig. 3. We find that with increasing atom numbers, states contain large N-GHZ components become increasingly localized to near the z - x equator plane. When $\phi = 0$, i.e. for a real qubit state, we always obtain a pure N-GHZ state. For $\phi \neq 0$, the optimal projection is such that θ should be as close as possible to the north/south pole, and the final state will be increasingly farther away from the N-GHZ state with increasing N . Although, as was shown earlier [12], one can simply apply a Raman coupling to rotate the collective interaction J_y^2 into the appropriate orthogonal direction to obtain a maximally entangled state.

5 Discussion and conclusion

For practical applications, our result relieves the restrictions on the N -qubit initial state to an arbitrary separate real pure state. It also relieves the odd/even-

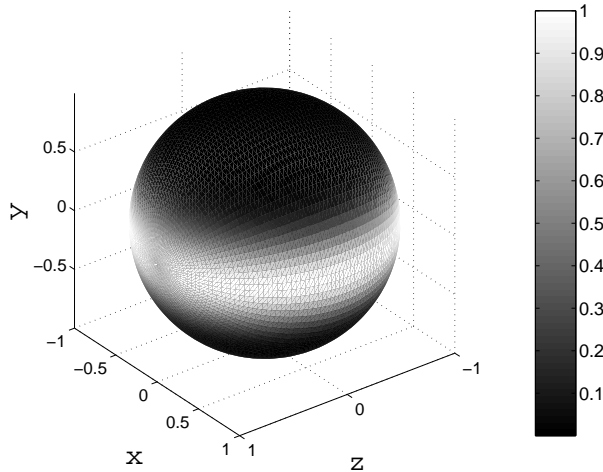


Fig. 3. The maximum N-GHZ state probability distribution for a condensate (of $N = 30$ atoms) in an initial state (θ, ϕ) on the Bloch sphere.

N constraint associated with N-qubit maximum entanglement using the collective J_y^2 -type interaction, i.e. one can simply affect the unitary evolution Eq. (3), or S_{ij} among all pairs irrespective whether the total number of qubit N being even or odd. Furthermore, when applied to ensembles of cold atoms, it now only requires the two state atoms interacting with each other with the same fixed collision strength, the atoms do not in principle have to be Bose condensed into the same spatial (condensate) mode. For that matter, they do not even have to be condensed. Although condensed atoms are believed to be less susceptible to decoherence.

In conclusion, we have provided an analytic approach for describing the controlled generation of N-GHZ states using a collective interaction $\sim J_y^2$. We have derived the maximum N-GHZ projection of the target state evolved from an arbitrary separable initial pure state and provided a simple procedure to characterize quantum correlations of the target state. We have developed a quantum secret sharing protocol based on the general N-GHZ states (19), thus illuminated their multi-party correlations despite of the unknown parameters θ_j 's. This is a rare example from an initial pure product state, whose generalized GHZ type quantum correlation due to the dynamic evolution with $\sim J_y^2$ can be efficiently characterized and understood at selected times. We believe our work will stimulate new experimental efforts to generate maximally entangled N-GHZ states.

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